

The universal definition of spin current

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Abstract

The spin current J_S , orbit angular momentum current J_L and total angular momentum current J_J in a dyad form have been universally defined according to quantum electrodynamics. Their conservation quantities and the continuity equations have been discussed in different cases. Non-relativistic approximation forms are deduced in order to explain their physical meanings, and to analyze some experiment results. The spin current of helical edge states in HgTe/CdTe quantum wells is calculated to demonstrate the properties of spin current on the two dimensional quantum spin-Hall system. A generalized spin-orbit coupling term in the semiconducting media is deduced based on the theory of the electrodynamics in the moving media. We recommend to use the effective total angular momentum current instead of the pure spin current to describe the polarizing distribution and transport phenomena in spintronic media.

INTRODUCTION

Spintronics [1, 2], a new sub-disciplinary field of condensed matter physics, has been regarded as bringing hope for a new generation of electronic devices. The advantages of spintronic devices include reducing the power consumption and overcoming the velocity limit of electric charge [1]. The two degrees of freedom of the spin enable to transmit more information in quantum computation and quantum information. In the past decade, many interesting phenomena emerged, moving the study of spintronics forward. The spin Hall effect predicts an efficient spin injection without the need of metallic ferromagnets [3], and can generate a substantial amount of dissipationless quantum spin current in a semiconductor [4]. All these provide the fundamental on designing spintronic devices, such as spin transistors [5] that were predicted several years ago. Experiment progresses have also been made in recent years [6, 7].

Since Rashba stated problems inherent in the theory of transport spin currents driven by external fields and gave his definition on the spin current tensor J_{ij} [8], there were several works on how to define the spin current in different cases. Sun et al. suggested that there was no need to modify the traditional definition on the spin current, but an additional term which describes the spin rotation should be included in the previous commonly accepted definition [9, 10]. A modified definition given by Shi [11] solved the conservation problem of the traditional spin current in the system Hamiltonian. His definition ensured an equilibrium thermodynamics theory built on spintronics, in accordance with other traditional transport theory, for instance, the Onsager relation.

Spin Hall effect, a vital phenomenon induced by spin-orbit coupling, has been extensively studied for years, although the microscopic origins of the effect are still being argued. Hirsch et al. [12] referred that anisotropic scattering by impurities will lead to the spin Hall effect, while an intrinsic cause of spin Hall effect was proposed by Sinova et al. [3]. Both theoretical and experimental work reported recently demonstrated the achievements of spin polarization in semiconductors [13–15].

In this letter, the spin current J_s , orbit angular momentum (OAM) current J_L and the total angular momentum (TAM) J_J , as well as the corresponding continuity equations have been delivered. In our dyad form expressions, the velocity operator α and the spin operator Σ can well display the physical meaning of the spin current. In addition, the non-relativistic

approximation (NRA) expressions have been derived and the quantum effects have been predicted in our expressions, which can not be deduced from previous works. Its vital effect on the finite size effect of the spin current will be showed and calculated in Hg/CdTe system. We recommend to use the effective TAM J and its current J_J to replace the traditional spin S and spin current in spintronics.

THE ANGULAR MOMENTUM IN DYAD FORM

According to the quantum electrodynamics theory, the Lagrangian

$$\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int} \quad (1)$$

$$= \bar{\psi} (ic\gamma^\mu \partial_\mu - mc^2) \psi - \frac{1}{4} F_{\mu\nu}^2 - \bar{\psi} c\gamma^\mu \psi A_\mu \quad (2)$$

$$= \bar{\psi} (ic\gamma^\mu D_\mu - mc^2) \psi - \frac{1}{4} F_{\mu\nu}^2 \quad (3)$$

can be represented in two terms.

$$\mathcal{L}_e = \bar{\psi} (ic\gamma^\mu D_\mu - mc^2) \psi, \quad (4)$$

$$\mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu}^2$$

, and the corresponding Hamiltonian of \mathcal{L}_e is well-known as

$$\hat{H} = c (\vec{\alpha} \cdot \vec{\Pi}) + \beta mc^2 + V \quad (5)$$

According to the Noether's theorem, one can derive the following equation

$$\partial_\mu (J_J)_\mu = 0. \quad (6)$$

while the corresponding Noether current is

$$J_J = J_s + J_L$$

. Here, the spin current density J_s is expressed

$$(J_s)_{\alpha\beta}^\mu = \frac{1}{4} \bar{\psi} (\gamma_\mu \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \gamma_\mu) \psi, \quad (7)$$

and the OAM current J_L

$$(J_L)_{\alpha\beta}^\mu = x_\alpha T_{\beta\rho} + x_\beta T_{\alpha\rho}$$

with $T_{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma_{\mu}D_{\nu}\psi$. Here γ_{μ} is the Dirac Matrix, and $\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_{\alpha}, \gamma_{\beta}]$. There may be some differences for choosing other representations of Dirac Matrix and the deduction details of Eq.(7) are shown in Appendix.

The Lorentz invariance of the Lagrangian ensures the conservation of TAM current J_J of electrons. Eq.(6) shows that the spin current alone is not conserved, unless the orbital angular momentum is fixed

The dyad form in 3D space

It is necessary to bridge the definition of the spin current J_S with the traditional descriptions in spintronics. Using the operator $\hat{\Sigma}_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix}$ and $\hat{\alpha}_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}$, the Eq.(7) turns as follows (shown in Appendix)

$$\left(\hat{J}_S\right)_{\mu\nu} = \frac{i}{2}\psi^{\dagger}\left(\hat{\alpha}_{\mu}\hat{\Sigma}_{\nu}\right)\psi, \quad (8)$$

thus the spin current operator is

$$\vec{J}_S = \frac{i}{2}\left(\hat{\alpha}\hat{\Sigma}\right) \quad (9)$$

where $\hat{\alpha}$ and $\hat{\Sigma}$ are the velocity operator and spin operator in Dirac equation, respectively.

In the traditional definition [9], the spin current density operator $\frac{1}{2}(\hat{v}\hat{s} + \hat{s}\hat{v})$ (Here, $\hat{v} = \frac{\hat{p}}{m}$ or $\frac{\hat{\Pi}}{m}$) means the carriers with a spin \hat{s} flowing at a speed of \hat{v} . However, the spin is an intrinsic physical character in quantum theory. The traditional definition based on an analogy of the classical current can not accurately describe the spin current.

Firstly, in relativistic quantum mechanics, the physical meanings of the velocity operator $\hat{\alpha}$ has been clearly described. Also, it should be pointed out that, there is a relationship between the electric current and the spin (the spin current) in $\frac{1}{c}$ order (which is shown in the next section). The spin-orbit coupling effect demands to replace the momentum operator \hat{p} (or $\hat{\Pi}$) with the operator $\hat{\alpha}$.

Secondly, for the commutation relation of $\hat{\alpha}$ and $\hat{\Sigma}$,

$$\left[\hat{\alpha}, \hat{\Sigma}\right] = i\hat{\alpha}$$

the quantum effect of the definition is lost during the classical analogy, especially in dealing with the finite size effect of the spin current and describing the experiment results in spintronics.

Deriving the expression of the OAM current J_L and the TAM J_J is similar to that of spin current J_s :

$$(J_L)_{\mu\nu} = \alpha_\mu L_\nu \quad (10)$$

$$\vec{J}_J = \vec{J}_s + \vec{J}_L \quad (11)$$

where OAM operator $L_\gamma = \epsilon_\gamma^{\alpha\beta} x_\alpha \Pi_\beta$.

Angular momentum current of photons

Generating and manipulating the polarization of electrons (or the carriers) is vital for spintronics. The main method is by letting the electron absorb or emit photons, in order to change its spin state.

The Lagrangian for a Maxwell field is

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}^2.$$

The corresponding terms to describe the photon's spin current, the OAM current and the TAM current are

$$\vec{J}_s^p = [\vec{\nabla} \vec{A}] \times \vec{A}, \quad (12)$$

$$\vec{J}_L^p = \vec{r} \times \vec{T}, \quad (13)$$

$$\vec{J}_J^p = \vec{J}_s^p + \vec{J}_L^p \quad (14)$$

respectively. Here $T_{ij} = \frac{1}{2}\delta_{ij} (E_i E_i + H_i H_i) - E_i E_j - H_i H_j$.

Obviously, only the TAM current $J_J + J_J^p$ meet the continuity equation

$$\frac{\partial}{\partial t} (\vec{J} + \vec{J}^p) + \nabla \cdot (\vec{J}_J + \vec{J}_J^p) = 0$$

By choosing the TAM current J_J (without the photon field) or $J_J + J_J^p$ (in the general occasion), one can keep the traditional theory unchanged, like the Onsager relation and the conservation law, which are built on the equilibrium state theory.

THE NRA EXPRESSION

In order to easily discuss and describe the physical meanings of the current expression, it is necessary to have a non-relativistic form of spin current. After some tedious simplifications

(shown in Appendix), we derive the non-relativistic expression of the spin current, OAM current and TAM current.

$$\vec{J}_s = \left(\vec{\Pi} \vec{\sigma} + \vec{\sigma} \vec{\Pi} + i \left(\vec{\sigma} \times \vec{\Pi} \right) \vec{\sigma} + i \vec{\sigma} \left(\vec{\Pi} \times \vec{\sigma} \right) \right) \quad (15)$$

$$\vec{J}_L = \left(\vec{\Pi} \vec{L} + \vec{L} \vec{\Pi} + i \left(\vec{\sigma} \times \vec{\Pi} \right) \vec{L} + i \vec{L} \left(\vec{\Pi} \times \vec{\sigma} \right) \right) \quad (16)$$

$$\vec{J}_J = \left(\vec{\Pi} \vec{J} + \vec{J} \vec{\Pi} + i \left(\vec{\sigma} \times \vec{\Pi} \right) \vec{J} + i \vec{J} \left(\vec{\Pi} \times \vec{\sigma} \right) \right) \quad (17)$$

where two important relations

$$\chi = \frac{(\sigma \cdot \Pi)}{2mc} \phi = \left(1 - \frac{1}{8m^2 c^2} \right) \frac{(\sigma \cdot \Pi)}{2mc} \psi$$

$$\left(\vec{\sigma} \cdot \vec{A} \right) \left(\vec{\sigma} \cdot \vec{B} \right) = \vec{A} \cdot \vec{B} + i \vec{\sigma} \cdot \left(\vec{A} \times \vec{B} \right)$$

are used. The result is shown to be completely equivalent to Eq.(9),(10) and (11) up to the order of $\frac{1}{c}$. Obviously, not only the traditional term of the spin current, but the other term

$$i \left(\vec{\sigma} \times \vec{\Pi} \right) \vec{\sigma} + i \vec{\sigma} \left(\vec{\Pi} \times \vec{\sigma} \right) \quad (18)$$

also contributes to the spin current in the same order.

In quantum physics, there are some quantum effects that can not be analogized with the classical theory. The term (18) can only be described as "similar" as a kind of quantum rotation. In Sun's work [9], the extra term ω_s is used to describe spin rotation, because a complete description of vector current should include translation and rotation motions as the classical theory shows. Here, the term (18) in our work which is accurately deduced proves two important conclusions as follows: First, the traditional definition of spin current can not make the spin conserved, which has been widely accepted. Second, the term (18) is the origin of the so-called quantum rotation, and its contribution is the exact source of the similar term in Sun's paper [9].

More importantly, because the term(18), with an "i" in its coefficient, stands for its quantum effect that can not be analogized classically, it does not only contribute to the magnitude of the spin current in the same order compared with the traditional definition, but also predict some important effects like the Spin Hall effect.

The diagonal matrix element of J_s is

$$J_{ii}^s = \frac{1}{4mc} \left(\hbar \left(\phi \sigma \cdot \nabla \phi^\dagger - \phi^\dagger \sigma \cdot \nabla \phi \right) - \frac{2e}{c} \phi^\dagger \sigma \cdot A \phi \right),$$

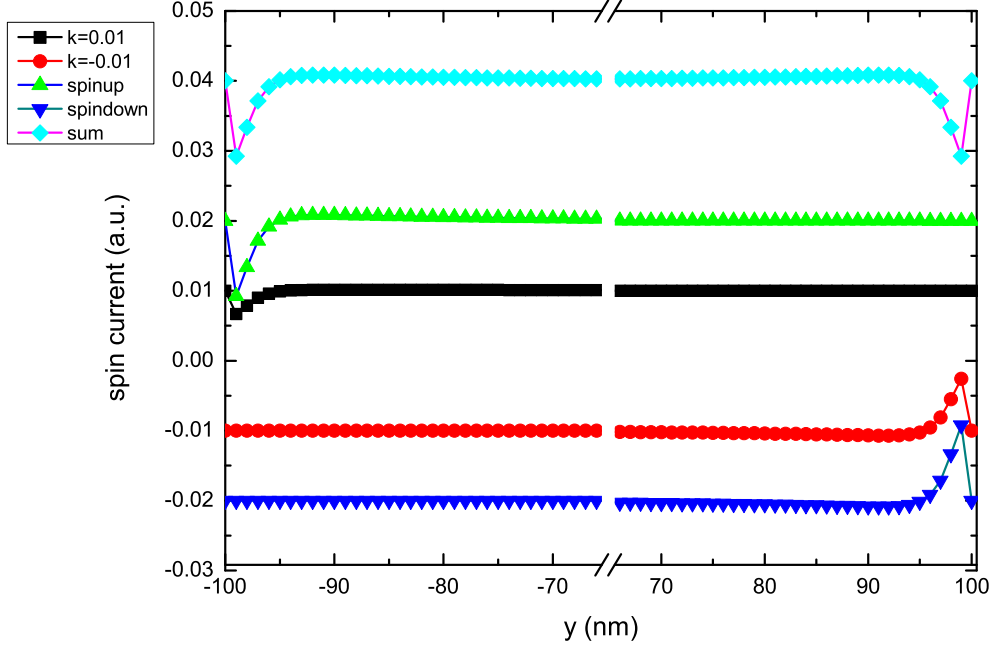


Figure 1. the spin current of $\Psi_{\uparrow+}(k_x, y)$ at $k = 0.01nm^{-1}$, $\Psi_{\downarrow-}(-k_x, y)$ at $k = -0.01nm^{-1}$, the spin up (solid/green), spin down, and the sum.

which is similar to the traditional definition of the spin current $\frac{1}{2}(vs + sv)$. While, the non-diagonal matrix element of the spin current that can be determined by

$$J_{ij} = \frac{1}{2mc} \epsilon_{ijk} j_k$$

with

$$j = \left(\frac{\hbar}{2m} (\phi(\nabla) \phi^\dagger - \phi^\dagger(\nabla) \phi) - \frac{e}{2m} \phi^\dagger \vec{A} \phi + \frac{\hbar}{2m} \nabla \times (\phi^\dagger \sigma \phi) \right)$$

is the exact matrix element of the current density operator in quantum electrodynamics.

Let's see an example of 2D HgTe/CdTe quantum well. First, we choose Kane model for semiconductors confining in a heterojunction of semiconductor HgTe/CdTe. The parameters are adopted from Ref [16].

Fig.1 shows the spin current of our definition. The wave functions $\Psi(k_x, y)$ are the edge states for $L = 200nm$. It is shown that the current exists not only in the bulk, but also on both edges (dependent on the spatial distribution parameters of the wave functions λ_1 , λ_2 and kinetic momentum k in Ref [16]), while, no spin current exists according to the traditional definition

$$\hat{J}_{yz} = \frac{1}{2} (\hat{v}_y \hat{s}_z + \hat{s}_z \hat{v}_y)$$

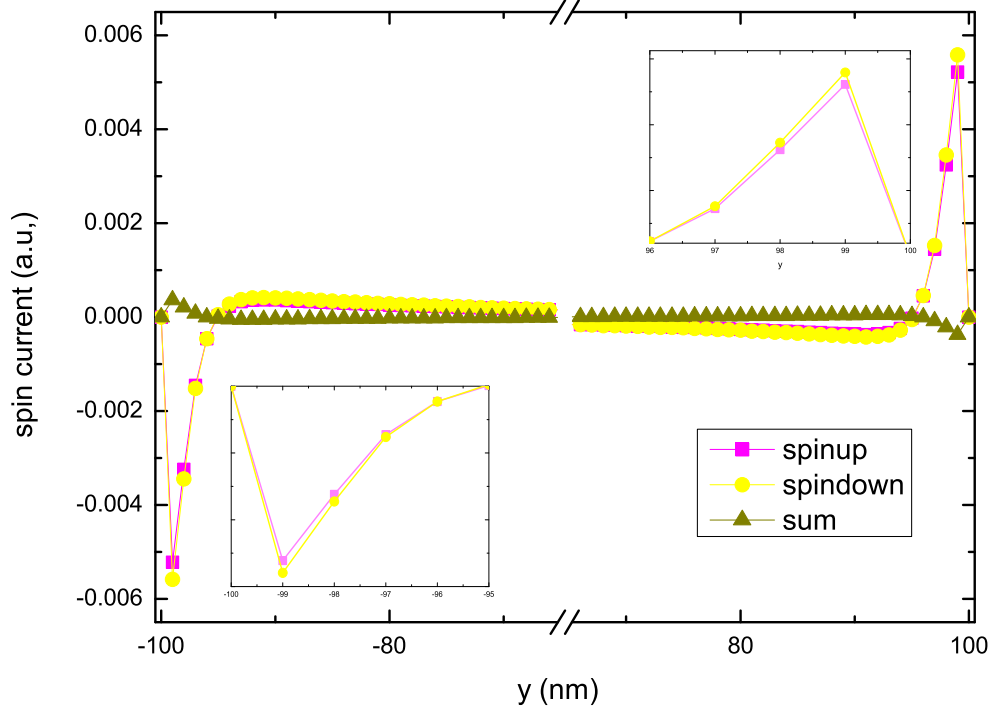


Figure 2. the spin current of $k = 0nm^{-1}$: the spin up, spin down, and the sum.

with $\hat{v}_y = -\frac{\hbar}{m_e}\partial_y$.

When $k = 0$, the spin current still exists in the surface, as shown in Fig2. This distinctive character other than the traditional electric current has been predicted in previous papers [1, 3, 4].

It should be pointed out that, because of the existence of the term (18), the surface effect of the spin current can be much more enhanced, since the quantum rotation is much stronger at the edge, and thus it contributes much more than the traditional definition of the spin current.

THE CONSERVATION AND THE CONTINUITY EQUATIONS

As pointed out, the conservation of spin current is a contradictory issue. Different conclusions have been drawn for taking different occasions into consideration. In non-relativistic quantum mechanics, the spin is a conserved quantity when the OAM is frozen. The continuity equation is

$$\frac{\partial}{\partial t}\vec{S} + \nabla \cdot \vec{J}_S = 0 \quad (19)$$

In Sun's work [9], he treated the spin as a classical vector, including two different motions. The translation motion can be described by the traditional definition of spin current, while the rotating motion can be described by the angular velocity operator $\vec{j}_\omega(r, t)$. Note that our extra term in Eq.(18) is the quantum origin of the rotation operator $\vec{j}_\omega(r, t)$. In the case when the OAM is not frozen (suitable for most spintronic systems), the continuity equation (19) turns into

$$\frac{\partial}{\partial t} \vec{J} + \nabla \cdot \vec{J}_J = 0.$$

The spin-orbit coupling effect results in that the spin is not a good quantum number any more. Because of the TAM J is the good quantum number, one can only choose the TAM \hat{J} and its corresponding current J_J to describe the transport phenomena. The theory of Quantum Electrodynamics points out that the electron's TAM can not stay conservation in the extra field. The Lorentz transformation of the system's Lagrangian gives out the continuity equation

$$\frac{\partial}{\partial t} (\vec{J} + \vec{J}^p) + \nabla \cdot (\vec{J}_J + \vec{J}_J^p) = 0 \quad (20)$$

This equation shows that the TAM of the system (the electrons and the photons) stays conservation. This is the physical meaning of Eq.(20). Eq.(20) can be written in another form

$$\frac{\partial}{\partial t} \vec{J} + \nabla \cdot \vec{J}_J = - \left(\frac{\partial}{\partial t} \vec{J}^p + \nabla \cdot \vec{J}_J^p \right)$$

The existence of \mathcal{L}_{int} enables the electrons and the photons to exchange angular momentum by some specific rules. This is exact the theoretical support on the experiments, namely by absorbing and emitting the photons, the electron's TAM can be changed. The change of photons' TAM state explains the corresponding polarizing phenomena in spintronics. Since the spin current by itself is not conserved, its rate equation can be derived using the Heisenberg equation of motion.

$$\begin{aligned} \frac{\partial}{\partial t} \vec{J}_S &= -\frac{i}{\hbar} [\vec{J}_S, H] \\ &= -\frac{i}{\hbar} \left(c \left(\vec{\alpha} \vec{\alpha} \times \vec{\Pi} + \vec{\Sigma} \times \vec{\Pi} \vec{\Sigma} \right) - i\beta mc^2 \vec{\alpha} \vec{\Sigma} \right) \end{aligned}$$

It shows obviously that not only the traditional term $\vec{\alpha} \cdot \vec{\Pi}$, but also βmc^2 has a commutation relation in J_S . A simple analogy with the classical theory can not describe the change of the spin current(or the TAM current).

THE TAM IN SEMICONDUCTORS

In the previous section, the differences between the spin current and the TAM current has been discussed. Besides the accuracy and quantum effect, the TAM current J_J has also other advantages to describe the polarizing distribution in spintronics. For III-V semiconductors, the quantum numbers of the wave functions are (j, j_z) , but not (s, m) , because the spin-orbit coupling plays an important role in deciding the energy band structures of the systems. Compared with the spin current, the TAM current is more accurate and meaningful, The physics qualities describing the transport phenomena are indirect spin-dependent, while they are the functions of the TAM current J and J_z . As usual, the media of the spintronic devices are mainly the magnetic or dilute magnetic semiconductors, which have a strong dielectric and magnetic polarization. According to the theory of electrodynamics in the media, especially considering the energy bands structure, the different Lande g of the angular momentum and the spin affect the spin-orbit coupling to some extent, and simply calculating the spin current using the traditional definition can not meet the need of describing and explaining the experimental results.

THE TAM IN MEDIA

The general spin-orbit coupling

The NRA of Dirac equation Eq.(5) can be written

$$H = H_1 + H_2 \quad (21)$$

where

$$H_1 = \frac{\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}{2m} - \frac{\vec{P}^3}{8m^3c^2} + eA_0 - \frac{e}{2mc}\vec{\sigma} \left(\nabla \times \vec{A}\right) + \frac{e}{8m^2c^2}\Delta A_0 \quad (22)$$

and

$$H_2 = -\frac{e}{4m^2c^2}\vec{\sigma} \left(\vec{E} \times \left(\vec{p} - \frac{e}{c}\vec{A}\right)\right) \quad (23)$$

The term H_2 is called the spin-orbit coupling, which is one of fundamentals of the spintronics. To study the carrier's transport properties, the electromagnetic susceptibility should be taken into calculation. In the case of the media having a relative velocity respect to the

carriers, the electromagnetic field in the polarized media interacting with the carriers is

$$\vec{D} = \epsilon \vec{E} + \frac{\epsilon\mu - 1}{c} \vec{v} \times \vec{H} \quad (24)$$

$$\vec{B} = \mu \vec{H} + \frac{\epsilon\mu - 1}{c} \vec{E} \times \vec{v} \quad (25)$$

where \vec{v} is the relative speed of the media in the field. By placing these relations into Eq.(21) and utilizing the relation

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r},$$

the Hamiltonian (up to $o(\frac{1}{m^2})$) turns to be

$$H = H'_1 + H'_2 \quad (26)$$

$$H'_1 = \frac{\hat{P}^2}{2m} - \frac{e\mu}{2mc} \left(\hat{\vec{L}} + \vec{\sigma} \right) \cdot \vec{H} + \frac{e^2}{2mc^2} \vec{A}^2 - \frac{\hat{P}^4}{8m^2c^2} + eA_0 + \frac{e}{8m^2c^2} \Delta A_0 \quad (27)$$

$$H'_2 = -\frac{e}{4m^2c^2} \left((2\epsilon\mu - 1) \vec{\sigma} + 2(\epsilon\mu - 1) \vec{L} \right) \cdot \left(\vec{E} \times \vec{\Pi} \right) \quad (28)$$

The spin-orbit coupling H_2 turns into a larger term H'_2 . According to the quantum electrodynamics, the spin-orbit coupling is induced by the electric field in which the electron moves at a speed of $\vec{\Pi}$ acting on the electron's spin.

$$\frac{e}{4m^2c^2} \vec{\sigma} \cdot \left(\vec{E} \times \vec{\Pi} \right) = \frac{e}{4m^2c^2} \frac{\partial V}{\partial r} \left(\vec{\sigma} \cdot \vec{L} \right) \quad (29)$$

When one considers the external fields in the solid-state media by the electromagnetic polarization for the moving carriers, one should include the OAM into calculation. This means that not only the spin, but also the OAM is coupled with the electric field. When $\epsilon\mu = 1$, the coupling term(28) turns back to be Eq.(29), the same as the traditional spin-orbit coupling. When $\epsilon\mu \gg 1$, however, the orbit angular accumulation affect the coupling term almost as well as that of the spin. Thus the OAM becomes crucial to describe the polarization of the system. According to the theory of the spin Hall effect, the carriers carrying different spins flow in the opposite directions. In our case, the carries with different angular momentums (j, j_z) flow in the different directions. The only difference is that the OAM is included in our model. It should be noticed that the condition $\epsilon\mu \gg 1$ usually holds in most semiconductors, like III-V compound semiconductors of GaAs, GaN etc. Thus,

$$\begin{aligned} \frac{e}{4m^2c^2} \left((2\epsilon\mu - 1) \vec{\sigma} + 2(\epsilon\mu - 1) \vec{L} \right) \cdot \left(\vec{E} \times \vec{\Pi} \right) &\doteq 2\epsilon\mu \left(g_s \vec{\sigma} + g_l \vec{L} \right) \cdot \left(\vec{E} \times \vec{\Pi} \right) \mu_B \\ &= 2\epsilon\mu \left(g_j \vec{J} \right) \cdot \left(\vec{E} \times \vec{\Pi} \right) \mu_B. \end{aligned} \quad (30)$$

According to the relation of the effective Lande g value and the effective mass, g in the Eq.(30) should be replaced by g^* in the semiconductors [17]. These imply that \vec{j} should replace the spin, as the physical quantity in more general cases.

THE DISCUSSION ON SOME EXPERIMENTS

Spin Hall effect

Zhang proposed a semi-classical Boltzmann-like equation to describe the distribution of the spins [18]. The similar behaviour can also be deduced from our definition, considering the finite size effects. In the system, the spin up current is

$$\langle J_s \rangle_{xz} = \langle \Psi_{\uparrow\pm} | J_s | \Psi_{\uparrow\pm} \rangle = J_t + J_e$$

The J_t and J_e are the traditional definition of the spin current and the extra term Eq.(18), respectively. As shown in the Appendix, J_t is proportional to k_x , namely

$$J_t^\pm = \mp C_h^\pm E_x$$

But J_e is independent on k_x , and is only as a function of the density distribution of electrons in y direction. namely

$$J_e^\pm = C_y^\pm E_y^\pm(y)$$

This, is a similar result compared to the Eqs (12) and (13) in Zhang's paper [18]. The spin accumulates in the y direction, which is exactly the same as that concluded from his anomalous Hall field. However, the spin diffusion is decided by the parameters ω and D in his conclusion. However, in our expression, while the spin diffusion are the corresponding parameters is dependent of the spatial distribution parameters λ_1, λ_2 [16] deduced from our definition of spin current.

The TAM Hall effect

Now we discuss about the spin Hall effect in GaAs bulk system with spin-orbit coupling effect on the energy band structure. According to Eq.(28), the Rashba effect can be written in $c_i k_i j_i$, $\Psi_{\frac{3}{2}, \frac{3}{2}}$, $\Psi_{\frac{3}{2}, \frac{1}{2}}$ and $\Psi_{\frac{1}{2}, \frac{1}{2}}$ (Ψ_{j, j_z}) accumulate to one edge while the $\Psi_{\frac{3}{2}, -\frac{3}{2}}$, $\Psi_{\frac{3}{2}, -\frac{1}{2}}$, $\Psi_{\frac{1}{2}, -\frac{1}{2}}$

on the other edge, namely the TAM j accumulates in both edges. It is easy to find that on both sides,

$$\left\langle \Psi(r) \left| \hat{J} \right| \Psi(r) \right\rangle = \sum_{j_z} \left\langle \frac{3}{2}, j_z \left| \hat{J} \right| \frac{3}{2}, j_z \right\rangle \neq 0. \quad (31)$$

According to the theory of Kerr rotation [19] $\theta_K = \theta'_K + i\theta''_K$, where

$$\theta'_K = -\frac{\omega_p^2}{2n(\epsilon_\perp - 1)} \Sigma_{ab} \frac{\beta_a}{\omega_{ab}} \frac{\Gamma_{ab}(\omega_{ab}^2 + \omega^2 + \Gamma)}{(\omega_{ab}^2 - \omega^2 + \Gamma_{ab}^2) + 4\omega^2\Gamma_{ab}^2} (f_{ab}^+ - f_{ab}^-)$$

$$\theta''_K = \frac{\omega_p^2\omega}{2n(\epsilon_\perp - 1)} \Sigma_{ab} \frac{\beta_a}{\omega_{ab}} \frac{(\omega_{ab}^2 - \omega^2 + \Gamma_{ab}^2)(f_{ab}^+ - f_{ab}^-)}{(\omega_{ab}^2 - \omega^2 + \Gamma_{ab}^2) + 4\omega^2\Gamma_{ab}^2}$$

where β_a is the probability density of the carriers occupying a energy level, ϵ_\perp is the dielectric quality, n is the refractive index, Γ_{ab} is the line width, ω is circular frequency of incident light and $\hbar\omega_{ab}$ is the energy gap. Obviously, the Kerr rotation angular is proportional to $(f_{ab}^+ - f_{ab}^-)$ whose expression is

$$f_{ab}^\pm = \frac{m\omega_{ab}}{\hbar e^2} |P_{ab}^\pm|^2 \quad (32)$$

$$P_{ab}^\pm = e \langle \Psi_a | x \pm iy | \Psi_b \rangle, \quad (33)$$

where Ψ_a is the ground state and Ψ_b is the excitation state. When $(f_{ab}^+ - f_{ab}^-) \neq 0$, the Kerr rotation occurs. As mentioned above, $\Psi_{\frac{3}{2}, \frac{3}{2}}, \Psi_{\frac{3}{2}, \frac{1}{2}}$ and $\Psi_{\frac{1}{2}, \frac{1}{2}}$ accumulate on one side, while $\Psi_{\frac{3}{2}, -\frac{3}{2}}, \Psi_{\frac{3}{2}, -\frac{1}{2}}$ and $\Psi_{\frac{1}{2}, -\frac{1}{2}}$ accumulate on the other side. Thus, $P_{hh}/P_{lh} = 16.4634$ at Γ point. As k varies, the ratio is influenced by Fermi surface according to the Rashba term. The accumulation of electrons in heavy hole bands attributes to the Kerr rotation. The TAM j accumulation gives the same image as the traditional spin Hall effect. Note that the spin does not accumulate actually, so the OAM plays an important role on the accumulation. More over, because the total angular momentum J offers more degrees of freedom, we can use it to transmit more information in the same condition. In summary, the spin-orbit coupling has been incorporated into the TAM j couples with the electric field. The OAM can be treated as the spin, especially in some system with a large $\epsilon\mu$. We recommend that the TAM j current replaces the spin current to describe the motion of the carriers with different angular momentum. The physical nature of polarization accumulation and the Kerr rotation has ben explained using our theory.

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Appendix

the definition of spin current

The Lagrangian of the system of $s = \frac{1}{2}$ is

$$\mathcal{L} = i\bar{\psi} (\gamma^\mu D_\mu - mc^2) \psi$$

According to the Noether's theorem, When $\psi' = \Lambda\psi = (1 - \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta})\psi$,

$$\begin{aligned} \delta(J_s)_{\alpha\beta}^\mu &= \frac{\partial\mathcal{L}}{\partial D_\mu\psi^i}\psi^j = i\bar{\psi}\gamma^\mu \left(1 - \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta}\right)\psi - i\bar{\psi} \left(1 + \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta}\right)\gamma^\mu\psi \\ &= -i\bar{\psi} \left(\gamma^\mu\frac{i}{4}\sigma_{\alpha\beta} + \frac{i}{4}\sigma_{\alpha\beta}\gamma^\mu\right)\epsilon^{\alpha\beta}\psi = \frac{1}{4}\bar{\psi}(\gamma^\mu\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\gamma^\mu)\epsilon^{\alpha\beta}\psi \end{aligned}$$

When $\psi' = \Lambda\psi = (1 - \frac{i}{2}\epsilon^{\mu\nu}x_\nu D_\mu)\psi$,

$$\begin{aligned} \delta(J_L)_{\alpha\beta}^\mu &= \frac{\partial\mathcal{L}}{\partial D_\mu\psi^i}\psi^j = i\bar{\psi}\gamma^\mu \left(1 - \frac{i}{2}\epsilon^{\alpha\beta}x_\alpha D_\beta\right)\psi - i\bar{\psi} \left(1 + \frac{i}{2}\epsilon^{\alpha\beta}x_\beta D_\alpha\right)\gamma^\mu\psi \\ &= \frac{1}{2}\bar{\psi}(x_\alpha\gamma^\mu D_\beta + x_\beta\gamma^\mu D_\alpha)\epsilon^{\alpha\beta}\psi \end{aligned}$$

Here, J_s^μ is the current operator of the spin s , J_L^μ is the current operator of the OAM .

$$(J_s)_{\alpha\beta}^\mu = \frac{1}{4}\bar{\psi}(\gamma^\mu\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\gamma^\mu)\psi$$

$$(J_L)_{\alpha\beta}^\mu = x_\alpha T_{\beta\rho} + x_\beta T_{\alpha\rho}$$

Here, $T_{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma_\mu D_\nu\psi$.

The dyad form

$$\begin{aligned} (J_s)_{\alpha\beta}^\mu &= \frac{1}{4}\bar{\psi}(\gamma^\mu\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\gamma^\mu)\psi = \frac{1}{4}\psi^\dagger\beta(\beta\alpha^\mu\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\beta\alpha^\mu)\psi \\ &= \frac{1}{4}\psi^\dagger(\alpha_\mu\Sigma_\gamma + \Sigma_\gamma\alpha_\mu)\psi = \frac{1}{2}\psi^\dagger(\alpha_\mu\Sigma_\gamma)\psi \end{aligned}$$

In Dirac representation, we have

$$\bar{\psi} = \psi^\dagger\beta, \gamma^\mu = \beta\alpha^\mu$$

$$\gamma_\alpha\gamma_\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma}\Sigma_\gamma$$

$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta] = \epsilon_{\alpha\beta\gamma}\Sigma_\gamma$$

The NRA form of spin current

The dyad form of spin current is

$$\vec{J}_s = \frac{i}{2} (\vec{\alpha} \vec{\Sigma})$$

and

$$H\psi = H \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

where $\chi = \frac{(\vec{\sigma} \cdot \Pi)}{2mc} \phi = \frac{\vec{\sigma} \cdot (-i\hbar \nabla - \frac{e}{c} \vec{A})}{2mc} \psi$. So

$$\begin{aligned} \langle \vec{J}_s \rangle &= \left\langle \psi^\dagger \left| \frac{i}{2} (\alpha \Sigma) \right| \psi \right\rangle = \frac{i}{2} \begin{pmatrix} \phi^* & \chi^* \end{pmatrix} \left(\begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}_\alpha \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}_\Sigma \right) \begin{pmatrix} \phi \\ \chi \end{pmatrix} \\ &= \frac{i}{2} \begin{pmatrix} \chi^* \vec{\sigma} & \phi^* \vec{\sigma} \end{pmatrix}_\alpha \begin{pmatrix} \vec{\sigma} \phi \\ \vec{\sigma} \chi \end{pmatrix}_\Sigma = \frac{1}{2} \left((\chi^* \vec{\sigma})_\alpha (\vec{\sigma} \phi)_\Sigma + (\phi^* \vec{\sigma})_\alpha (\vec{\sigma} \chi)_\Sigma \right) \\ &= \frac{i}{2} \left(\left(\frac{\vec{\sigma} \cdot (-i\hbar \nabla - \frac{e}{c} \vec{A})}{2mc} \psi^* \vec{\sigma} \right)_\alpha (\vec{\sigma} \psi)_\Sigma + (\psi^* \vec{\sigma})_\alpha \left(\vec{\sigma} \frac{\vec{\sigma} \cdot (-i\hbar \nabla - \frac{e}{c} \vec{A})}{2mc} \psi \right)_\Sigma \right) \\ &= \frac{i}{4mc} \left(\left(\vec{\sigma} \cdot (-i\hbar \nabla - \frac{e}{c} \vec{A}) \psi^* \vec{\sigma} \right)_\alpha (\vec{\sigma} \psi)_\Sigma + (\psi^* \vec{\sigma})_\alpha \left(\vec{\sigma} \cdot \vec{\sigma} \left(-i\hbar \nabla - \frac{e}{c} \vec{A} \right) \psi \right)_\Sigma \right) \quad (34) \end{aligned}$$

As

$$(\vec{\sigma} \cdot \vec{A}) (\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i \vec{\sigma} \cdot (\vec{A} \times \vec{B})$$

So

$$\begin{aligned} \vec{\sigma} (\vec{\sigma} \cdot \vec{B}) &= \vec{B} + i (\vec{B} \times \vec{\sigma}) \\ (\vec{\sigma} \cdot \vec{A}) \vec{\sigma} &= \vec{A} + i (\vec{\sigma} \times \vec{A}) \end{aligned}$$

The Eq. (34) turns to be

$$\begin{aligned} \langle \vec{J}_s \rangle &= \frac{i}{4mc} \left(\left(\vec{\sigma} \cdot (-i\hbar \nabla - \frac{e}{c} \vec{A}) \psi^* \vec{\sigma} \right)_\alpha (\vec{\sigma} \psi)_\Sigma + (\psi^* \vec{\sigma})_\alpha \left(\vec{\sigma} \cdot (-i\hbar \nabla - \frac{e}{c} \vec{A}) \psi \right)_\Sigma \right) \\ \langle J_s \rangle_{ij} &= \begin{cases} \frac{1}{4mc} \left(\hbar (\phi \vec{\sigma} \cdot \nabla \phi^\dagger - \phi^\dagger \vec{\sigma} \cdot \nabla \phi) - \frac{2e}{c} \phi^\dagger \vec{\sigma} \cdot \vec{A} \phi \right)_i & \text{when } i = j \\ \frac{1}{2mc} \epsilon_{ijk} \left(\frac{\hbar}{2m} (\phi (\nabla) \phi^\dagger - \phi^\dagger (\nabla) \phi) - \frac{e}{2m} \phi^\dagger \vec{A} \phi + \frac{\hbar}{2m} \nabla \times (\phi^\dagger \vec{\sigma} \phi) \right)_k & \text{when } i \neq j \end{cases} \end{aligned}$$

The NRA expressions of the OAM current and the TAM current are similar, except that the $(\sigma)_\Sigma$ should be changed into the operators L and J , respectively.

The momentum current of Photon

The Lagrangian of the system of $s = 1$ is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A^\nu \partial_\nu A^\mu$$

. Similar to the progress in Appendix 1, $\delta A^i = \epsilon_{jk}^i A^j \theta^k$, and according to the Noether's theorem, we have

$$j^0 = ((J_L^p)_k + (J_S^p)_k) \theta^k$$

. The OAM current and the spin current are

$$J_L^p = \epsilon_{ijk} x^i P^j$$

$$J_S^p = \frac{\partial \mathcal{L}}{\partial \partial_0 A^i} \epsilon_{jk}^i A_j$$

where $P^i = \frac{\partial \mathcal{L}}{\partial \partial_0 A^\mu} A^\mu - \mathcal{L} g^{0i}$.

The motion equations of angular momentum currents

According to the Heisenberg equation, we have

$$\begin{aligned} \frac{\partial}{\partial t} (J_S)_{\mu\nu} &= -\frac{i}{\hbar} \left[(J_S)_{\mu\nu}, H \right] = -\frac{i}{\hbar} \left[\frac{1}{2} (\alpha_\mu \Sigma_\nu), c \left(\vec{\alpha} \cdot \vec{\Pi} \right) + \beta m c^2 + V \right] \\ &= -\frac{i}{\hbar} \left[\frac{1}{2} (\alpha_\mu \Sigma_\nu), c \left(\vec{\alpha} \cdot \vec{\Pi} \right) \right] - \frac{i}{\hbar} \left[\frac{1}{2} (\alpha_\mu \Sigma_\nu), \beta m c^2 \right] \end{aligned} \quad (35)$$

Because of the relations

$$\begin{aligned} [\alpha_i, \alpha \cdot \Pi] &= [\alpha_i, \alpha_i \cdot \Pi_i + \alpha_j \cdot \Pi_j + \alpha_k \cdot \Pi_k] = [\alpha_i, \alpha_j \cdot \Pi_j + \alpha_k \cdot \Pi_k] \\ &= 2i (\Pi_j \Sigma_k - \Pi_k \Sigma_j) \end{aligned}$$

$$\begin{aligned} [\Sigma_i, \alpha \cdot \Pi] &= [\Sigma_i, \alpha_i \cdot \Pi_i + \alpha_j \cdot \Pi_j + \alpha_k \cdot \Pi_k] = [\Sigma_i, \alpha_j \cdot \Pi_j + \alpha_k \cdot \Pi_k] \\ &= 2i (\Pi_j \alpha_k - \Pi_k \alpha_j) \end{aligned}$$

$$[\alpha, \beta] = -2\beta\alpha$$

namely,

$$\begin{aligned} \left[\frac{1}{2} (\alpha_\mu \Sigma_\nu), c \left(\vec{\alpha} \cdot \vec{\Pi} \right) \right] &= \left[\frac{1}{2} \alpha_\mu, c \left(\vec{\alpha} \cdot \vec{\Pi} \right) \right] \Sigma_\nu + \alpha_\mu \left[\frac{1}{2} \Sigma_\nu, c \left(\vec{\alpha} \cdot \vec{\Pi} \right) \right] \\ &= -ic \left((\Sigma \times \Pi)_\mu \Sigma_\nu + \alpha_\mu (\alpha \times \Pi)_\nu \right) \end{aligned}$$

$$\begin{aligned}\left[\frac{1}{2}(\alpha_\mu \Sigma_\nu), \beta m c^2\right] &= \left[\frac{1}{2}\alpha_\mu, \beta m c^2\right] \Sigma_\nu + \alpha_\mu \left[\frac{1}{2}\Sigma_\nu, \beta m c^2\right] \\ &= -\beta \alpha_\mu m c^2 \Sigma_\nu\end{aligned}$$

the Eq.(35) turns into

$$\begin{aligned}\frac{\partial}{\partial t}(J_S)_{\mu\nu} &= -\frac{i}{\hbar} \left[(J_S)_{\mu\nu}, H\right] \\ &= -\frac{1}{\hbar} \left(c \left(\alpha_\mu (\alpha \times \Pi)_\nu + (\Sigma \times \Pi)_\mu \Sigma_\nu\right) - i\beta m c^2 \alpha_\mu \Sigma_\nu\right)\end{aligned}$$

Spin Hall effect in finite size effect

For the edge sates $\Psi_{\uparrow+}$

$$\begin{aligned}J_t^+ &= \tilde{c}_+^* e^{-ik_x x} (f_+ + \gamma_{k_x}^+ f_-, \eta_1^+ f_- + \gamma_{k_x}^+ \eta_2^+ f_+) \frac{1}{2} [(k_x \uparrow \uparrow \sigma_z) + h.c.] \\ &\quad \tilde{c}_+ e^{ik_x x} (f_+ + \gamma_{k_x}^+ f_-, \eta_1^+ f_- + \gamma_{k_x}^+ \eta_2^+ f_+)^T \\ &= k_x \left[(f_+ + \gamma_{k_x}^+ f_-)^2 - (\eta_1^+ f_- + \gamma_{k_x}^+ \eta_2^+ f_+)^2 \right]\end{aligned}$$

$$\begin{aligned}J_e^+ &= \tilde{c}_+^* e^{-ik_x x} (f_+ + \gamma_{k_x}^+ f_-, \eta_1^+ f_- + \gamma_{k_x}^+ \eta_2^+ f_+) \frac{1}{2} [(i\sigma \times k \uparrow \uparrow \sigma_z) + h.c.] \\ &\quad \tilde{c}_+ e^{ik_x x} (f_+ + \gamma_{k_x}^+ f_-, \eta_1^+ f_- + \gamma_{k_x}^+ \eta_2^+ f_+)^T \\ &= \tilde{c}_+^* e^{-ik_x x} (f_+ + \gamma_{k_x}^+ f_-, \eta_1^+ f_- + \gamma_{k_x}^+ \eta_2^+ f_+) \frac{1}{2} [(-\sigma_z \partial_y \uparrow \uparrow \sigma_z) + h.c.] \\ &\quad \tilde{c}_+ e^{ik_x x} (f_+ + \gamma_{k_x}^+ f_-, \eta_1^+ f_- + \gamma_{k_x}^+ \eta_2^+ f_+)^T \\ &= (f_+ + \gamma_{k_x}^+ f_-) (f'_+ + \gamma_{k_x}^+ f'_-) + (\eta_1^+ f_- + \gamma_{k_x}^+ \eta_2^+ f_+) (\eta_1^+ f'_- + \gamma_{k_x}^+ \eta_2^+ f'_+)\end{aligned}$$

For the edge sates $\Psi_{\uparrow-}$

$$\begin{aligned}J_t^- &= k_x \left[(f_- + \gamma_{k_x}^- f_+)^2 - (\eta_2^- f_+ + \gamma_{k_x}^- \eta_1^- f_-)^2 \right] \\ J_e^- &= (f_- + \gamma_{k_x}^- f_+) (f'_- + \gamma_{k_x}^- f'_+) + (\eta_2^- f_+ + \gamma_{k_x}^- \eta_1^- f_-) (\eta_2^- f'_+ + \gamma_{k_x}^- \eta_1^- f'_-)\end{aligned}$$

The Spin-Orbit coupling in media

According to the Maxwell equations in the media

$$\begin{aligned}\vec{D} &= \epsilon \vec{E} + \frac{\epsilon\mu - 1}{c} \vec{v} \times \vec{H} \\ \vec{B} &= \mu \vec{H} + \frac{\epsilon\mu - 1}{c} \vec{E} \times \vec{v}\end{aligned}$$

the first term in the NRA of Dirac equation turns to be

$$\begin{aligned}
H_1 &= \frac{\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}{2m} - \frac{\vec{P}^3}{8m^3c^2} + eA_0 - \frac{e}{2mc}\vec{\sigma}\left(\nabla \times \vec{A}\right) + \frac{e}{8m^2c^2}\Delta A_0 \\
&= \frac{\left(\vec{P} - \frac{e}{2c}\left(\vec{B} \times \vec{r}\right)\right)^2}{2m} - \frac{\vec{P}^3}{8m^3c^2} + eA_0 - \frac{e}{2mc}\vec{\sigma}\vec{B} + \frac{e}{8m^2c^2}\Delta A_0 \\
&= \frac{\vec{P}^2}{2m} - \frac{e}{2mc}\left(\vec{L} \cdot \vec{B}\right) + \frac{e^2}{8mc^2}\left(\vec{B} \times \vec{r}\right)^2 - \frac{\vec{P}^3}{8m^3c^2} + eA_0 - \frac{e}{2mc}\vec{\sigma}\vec{B} + \frac{e}{8m^2c^2}\Delta A_0 \\
&= \frac{\vec{P}^2}{2m} - \frac{e\mu}{2mc}\left(\vec{L} + \vec{\sigma}\right) \cdot \vec{H} + \frac{e^2}{2mc^2}\vec{A}^2 - \frac{\vec{P}^4}{8m^2c^2} + eA_0 + \frac{e}{8m^2c^2}\Delta A_0 \\
&= H'_1 - \frac{e(\epsilon\mu - 1)}{2mc^2}\left(\vec{L} + \vec{\sigma}\right)\left(\vec{E} \times \vec{\Pi}\right)
\end{aligned}$$

So

$$H_2 - \frac{e(\epsilon\mu - 1)}{2mc^2}\left(\vec{L} + \vec{\sigma}\right)\left(\vec{E} \times \vec{\Pi}\right) = -\frac{e}{4m^2c^2}\left((2\epsilon\mu - 1)\vec{\sigma} + 2(\epsilon\mu - 1)\vec{L}\right) \cdot \left(\vec{E} \times \vec{\Pi}\right)$$

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- [1] I. Žutić, J. Fabian, and S. D. Sarma, Reviews of Modern Physics, **76**, 323 (2004).
- [2] S. A. Wolf, D. D. Awschalom, R. A. Buhrman, J. M. Daughton, S. von Molnar, M. L. Roukes, A. Y. Chtchelkanova, and D. M. Treger, Science, **294**, 1488 (2001).
- [3] J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Phys. Rev. Lett., **92**, 126603 (2004).
- [4] S. Murakami, N. Nagaosa, and S. Zhang, Science, **301**, 1348 (2003).
- [5] S. Datta and B. Das, Appl. Phys. Lett., **56**, 665 (1990).
- [6] Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, Science, **306**, 1910 (2004).
- [7] S. Matsuzaka, Y. Ohno, and H. Ohno, Physical Review B, **80**, 241305 (2009).
- [8] E. I. Rashba, Physical Review B, **68**, 241315 (2003).
- [9] Q. feng Sun and X. C. Xie, Physical Review B, **72**, 245305 (2005).
- [10] Q. feng Sun, X. C. Xie, and J. Wang, Physical Review B, **77**, 035327 (2008).
- [11] J. Shi, P. Zhang, D. Xiao, and Q. Niu, Phys. Rev. Lett., **96**, 076604 (2006).
- [12] J. E. Hirsch, Phys. Rev. Lett., **83**, 1834 (1999).

- [13] Y. Ohno, D. Young, B. Beschoten, F. Matsukura, H. Ohno, and D. Awschalom, *Nature*, **402**, 790 (1999).
- [14] N. Tombros, C. Jozsa, M. Popinciuc, H. T. Jonkman, and B. J. van Wees, *Nature*, **448**, 571 (2007).
- [15] S. Valenzuela and M. Tinkham, *Nature*, **442**, 176 (2006).
- [16] B. Zhou, H. Lu, R. Chu, S. Shen, and Q. Niu, *Phys. Rev. Lett.*, **101**, 246807 (2008).
- [17] K. Shen, M. Q. Weng, and M. W. Wu, *Journal of Applied Physics*, **104**, 063719 (2008).
- [18] S. Zhang, *Phys. Rev. Lett.*, **85**, 393 (2000).
- [19] E. Condon and G. H. Shortley, *The theory of atomic spectra*, repr. ed. (Univ. Press, Cambridge, 1977).